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Stochastic approach for the subordination in Bochner sense

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It is possible to construct (cf [1] and [2]) a positive process $(Y_{\psi,t})$ indexed by Bernstein functions and time which, for fixed t , is Markovian with respect to the composition of Bernstein functions, and for fixed ψ , is the subordinator associated with the Bernstein function ψ . In the same manner a realization $(X_{Y_{\psi,t}})$ can be obtained of all the subordinated processes of a Markov process (X_t) .

This probabilistic interpretation of the initial idea of Bochner allows to construct martingales and to apply stochastic calculus to questions related to symbolic calculus for operators semi-groups.

We study here a particular branch of the subordination process : The homographic branch which has the advantage of being connected with some works on positive infinitely divisible diffusions and brings on them a different point of view. Nevertheless, what follows would apply, except technical difficulties, to any branch of the subordination process.

It is easy to see that the functions

$$f_a(x) = \frac{xe^a}{1 + x(e^a - 1)}, \quad a \geq 0, x \geq 0$$

satisfy $f_a \circ f_b = f_{a+b}$ and are Bernstein functions (cf [3]) associated with the subordinators

$$(1) \quad Y_t^a = (e^a - 1) \sum_{k=1}^{N_t/(1-e^a)} E_k$$

where (N_t) is a standard Poisson process and the E_k 's are i.i.d. exponential random variables independent of N . The Lévy measure of $(Y_{a,t})$ is therefore

$$\nu_a(dy) = \frac{1}{(e^a - 1)^2} e^{-y/(e^a - 1)} dy \quad \text{on } \mathbb{R}_+.$$

The relation $P_a u(x) = \mathbb{E}[u(Y_x^a)]$ for $u : \mathbb{R}_+ \mapsto \mathbb{R}_+$ defines a Markovian semi-group $(P_a)_{a \geq 0}$ with generator $Au(x) = xu'(x) + xu''(x)$ which is the transition semi-group of the diffusion $(Z_a)_{a \geq 0}$ solution to the sde

$$(2) \quad dZ_a = \sqrt{2Z_a} dB_a + Z_a da, \quad Z_0 = z \geq 0.$$

We now define a two parameters process $(Y_{a,t})_{a \geq 0, t \geq 0}$ by choosing for $a_1 < a_2 < \dots < a_n$ the joint law of the processes

$$(Y_{a_1,t})_{t \geq 0}, \dots, (Y_{a_n,t})_{t \geq 0}$$

to be that of the process

$$\left(Y_t^{a_1}, Y_{Y_t^{a_1}}^{a_2-a_1}, \dots, Y_{Y_t^{a_{n-1}-a_{n-2}}}^{a_n-a_{n-1}}, \dots, Y_t^{a_1} \right)_{t \geq 0}$$

where $(Y_t^{a_1})_{t \geq 0}, (Y_t^{a_2-a_1})_{t \geq 0}, \dots, (Y_t^{a_n-a_{n-1}})_{t \geq 0}$ are independent subordinators of type (1).

A version of the process $(Y_{a,t})$ may be chosen such that $Y_t = Y_{.,t}$ be right continuous and increasing with values in $\mathcal{C}(\mathbb{R}_+, \mathbb{R}_+)$ and with independent stationary increments, and for fixed t $(Y_{a,t})_{a \geq 0}$ has the same law as (Z_a) for $Z_0 = t$.

Because of the formula $\nu_a P_b = \nu_{a+b}$, we can construct the process $(Z_a)_{a \geq 0}$ with the entrance rule (ν_a) what defines a positive σ -finite measure m on the space $\mathcal{C}(\mathbb{R}_+^*, \mathbb{R}_+)$ as the ‘law’ of the process (Z_a) under the entrance law (ν_a) .

This measure m is the Lévy measure of the process $(Y_t)_{t \geq 0}$. This can be seen in the following way. Let μ be a positive measure with compact support on \mathbb{R}_+ , let us put $\langle Y_t, \mu \rangle = \int Y_{a,t} \mu(da)$. The relation

$$(3) \quad \mathbb{E} e^{-\langle Y_t, \mu \rangle} = \exp \int (e^{\int Z_a \mu(d\alpha)} - 1) dm$$

is easy to prove when μ is a weighted sum of Dirac masses by the computation

$$\mathbb{E} e^{-\langle Y_t, \lambda \varepsilon_a \rangle} = \mathbb{E} e^{-\lambda Y_{a,t}} = e^{-t f_a(\lambda)} = e^{t \int (e^{-\lambda y} - 1) \nu_a(dy)}$$

noting that ν_a is the law of Z_a under m . And for general μ (3) is obtained by weak limit.

The law of the subordinators $\langle Y_t, \mu \rangle$, may be studied by using the fact that if we decompose μ into

$$\mu = 1_{[0,x]} \cdot \mu + 1_{(x,\infty)} \cdot \mu = \mu_1 + \mu_2$$

the following representation holds

$$\langle Y_t, \mu \rangle = \langle Y_t, \mu_1 \rangle + \int Y_{Y_{x,t}}^{b-x} \mu_2(db)$$

where $(Y_t^\beta)_{\beta \geq 0, t \geq 0}$ is independent of $(Y_{a,t}, a \leq x, t \geq 0)$.

One obtains for $\lambda \geq 0$,

$$\mathbb{E} \exp\{-\lambda \langle Y_t, \mu \rangle\} = \exp\left\{-t\left(\frac{1}{2} - g'_x(0, \lambda)\right)\right\}$$

where g is the positive solution decreasing in x of

$$\begin{cases} g''_{x^2} = g \cdot (\frac{1}{4} + \lambda \cdot \mu) \\ g(0, \lambda) = 1 \end{cases}$$

and if we put $\xi(x, \lambda) = \frac{1}{2} - \frac{g'_x(x, \lambda)}{g(x, \lambda)}$, the process

$$M_{a,t} = \exp\{-\xi(a, \lambda)Y_{a,t} + \xi(a, \lambda)t - \lambda \int_0^a Y_{\alpha,t} \mu(d\alpha)\}$$

is a two parameters martingale for the filtration of $(Y_{a,t})$.

A similar study may be performed with the family $g_a(x) = \frac{x}{1+ax}$ which corresponds to the diffusion

$$(4) \quad Z_a = z + \int_0^a \sqrt{2Z_b} dB_b.$$

which is the square of a Bessel process of exponent 0, cf [4].

These results were obtained by Pitman and Yor [5], [6], by interpreting the process $Y_{a,t}$ of the case (4) as $\ell_{\tau_t}^a$ where ℓ_t^a is the family of brownian local times and $\tau_t = \inf\{s : \ell_s^0 = t\}$, and using the Ray-Knight theorems. In this framework the measure m appears to be the image of the Ito measure of the Brownian excursions.

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